Surds

How do surds behave?

TOP TIP

$\sqrt{4}+\sqrt{9}=2+3=5$

$\sqrt{4+9}=\sqrt{13}\neq 5$

Compare:	A quick calculator check gives:	
$\sqrt{2+3}$ and $\sqrt{2}+\sqrt{3}$	$\sqrt{2+3}=\sqrt{5}=2\cdot23\ldots$ $\sqrt{2}+\sqrt{3}=1\cdot41\ldots+1\cdot73\ldots=3\cdot14\ldots$	Not the same
$\sqrt{2\times3}$ and $\sqrt{2}\times\sqrt{3}$	$\sqrt{2\times3}=\sqrt{6}=2\cdot44\ldots$ $\sqrt{2}\times\sqrt{3}=1\cdot41\ldots\times1\cdot73\ldots=2\cdot44\ldots$	They might be the same!
$\sqrt{\frac{2}{3}}$ and $\frac{\sqrt{2}}{\sqrt{3}}$	$\sqrt{\frac{2}{3}}=0\cdot81\ldots$ $\frac{\sqrt{2}}{\sqrt{3}}=\frac{1\cdot41\ldots}{1\cdot73\ldots}=0\cdot81\ldots$	They might be the same!

The general rules that are true are:

$\sqrt{a\times b}=\sqrt{a}\times\sqrt{b}$ and $\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}$ These can be proved.

Note: $\sqrt{a}^2=\sqrt{a}\times\sqrt{a}=\sqrt{a\times a}=\sqrt{a^2}=a$

Simplifying surds

TOP TIP

When simplifying a surd look for the highest square number factor.

The square numbers

1 16 36 81 4 9 49 100 25 64

To simplify $\sqrt{108}$ find a factor of 108 that is a square number, e.g. 9. You know $108 = 9 \times 12$.

So $\sqrt{108}=\sqrt{9\times12}=\sqrt{9}\times\sqrt{12}=3\times\sqrt{12}$.

Just as $3\times x$ is written $3x$ so $3\times\sqrt{12}$ is written $3\sqrt{12}$.

Now 12 has a square factor of 4.

So $3\sqrt{12}=3\times\sqrt{4\times3}=3\times\sqrt{4}\times\sqrt{3}$

$=3\times2\times\sqrt{3}=6\times\sqrt{3}=6\sqrt{3}$

Here is a quicker method:

$\sqrt{108}=\sqrt{36\times3}=\sqrt{36}\times\sqrt{3}=6\times\sqrt{3}=6\sqrt{3}$

When the number under the root sign is as small as possible (no more square factors other than 1) you have fully simplified the surd.

Examples

Simplify: (a) $\sqrt{96}$ (b) $\sqrt{\frac{81}{100}}$

(a) $\sqrt{96}=\sqrt{16\times6}=\sqrt{16}\times\sqrt{6}=4\times\sqrt{6}=4\sqrt{6}$

(b) $\sqrt{\frac{81}{100}}=\frac{\sqrt{81}}{\sqrt{100}}=\frac{9}{10}=0\cdot9$

Preparation for Higher

Surds

1. Which of these numbers are surds:

$\sqrt{16}, \sqrt{65}, \sqrt{1}, \sqrt[3]{8}, \sqrt[3]{9}, \sqrt[3]{1}, \sqrt{50}, \sqrt[3]{33}, \sqrt[3]{27}, \sqrt{5}, \sqrt{1000}, \sqrt[3]{-1000}$

2. Find the exact solution of each equation.

(a) $x^2 - 5 = 9$ (b) $x^2 + 6 = 36$ (c) $x^3 - 4 = 60$
(d) $x^2 + 11 = 12$ (e) $x^3 - 13 = 26$ (f) $x^3 + 20 = 19$

Simplifying surds

1. Express in simplest form:

(a) $\sqrt{12}$ (b) $\sqrt{20}$ (c) $\sqrt{27}$ (d) $\sqrt{32}$
(e) $\sqrt{45}$ (f) $\sqrt{48}$ (g) $\sqrt{50}$ (h) $\sqrt{63}$
(i) $\sqrt{75}$ (j) $\sqrt{44}$ (k) $\sqrt{98}$ (l) $\sqrt{500}$
(m) $5\sqrt{8}$ (n) $3\sqrt{18}$ (o) $4\sqrt{200}$ (p) $3\sqrt{1000}$

2. Simplify:

(a) $7\sqrt{2} + 3\sqrt{2}$ (b) $9\sqrt{5} - 5\sqrt{5}$ (c) $\sqrt{3} + 6\sqrt{3}$
(d) $4\sqrt{7} - \sqrt{7}$ (e) $9\sqrt{10} - 9\sqrt{10}$ (f) $\sqrt{5} - 8\sqrt{5}$
(g) $3\sqrt{2} - \sqrt{2} + 7\sqrt{2}$ (h) $\sqrt{7} + \sqrt{5} + 2\sqrt{7}$ (i) $2\sqrt{10} - 10\sqrt{2}$
(j) $25/ + 3\sqrt{2} - 2\sqrt{5} - \sqrt{2}$ (k) $-4\sqrt{11} + 8\sqrt{10} - 2\sqrt{11} - 2\sqrt{10}$

3. Solve these equations, where necessary leaving the answer as a surd in its simplest form.

(a) $x^2 + 8 = 36$ (b) $x^2 - 15 = 60$ (c) $\frac{1}{2}x^2 + 2 = 51$
(d) $x^2 - 147 = 0$ (e) $x^3 + 12 = 4$ (f) $x^3 - 5 = 49$

Multiplication of Surds

1. Simplify:

(a) $\sqrt{3} \times \sqrt{3}$ (b) $\sqrt{7} \times \sqrt{7}$ (c) $\sqrt{2a} \times \sqrt{2a}$
(d) $\sqrt{4} \times \sqrt{3}$ (e) $\sqrt{9} \times \sqrt{2}$ (f) $\sqrt{3} \times \sqrt{25}$
(g) $\sqrt{2} \times \sqrt{8}$ (h) $\sqrt{7} \times \sqrt{3}$ (i) $\sqrt{11} \times \sqrt{2}$
(j) $\sqrt{2} \times \sqrt{8}$ (k) $\sqrt{12} \times \sqrt{3}$ (l) $\sqrt{2} \times \sqrt{50}$
(m) $\sqrt{2} \times \sqrt{10}$ (n) $\sqrt{3} \times \sqrt{6}$ (o) $\sqrt{8} \times \sqrt{12}$
(p) $\sqrt{10} \times \sqrt{20}$ (q) $3\sqrt{2} \times 5\sqrt{2}$ (r) $3\sqrt{5} \times 5\sqrt{3}$

2. Simplify

(a) $\sqrt{2}(1 + \sqrt{2})$ (b) $\sqrt{3}(\sqrt{3} - 1)$
(c) $(1 + \sqrt{5})\sqrt{5}$ (d) $\sqrt{7}(5 + \sqrt{7})$
(e) $\sqrt{2}(3 - 2\sqrt{2})$ (f) $(3\sqrt{5} - 2)\sqrt{5}$
(g) $(\sqrt{3} + 1)(\sqrt{3} - 1)$ (h) $(\sqrt{5} - 2)(\sqrt{5} + 2)$
(i) $(3 + \sqrt{7})(3 - \sqrt{7})$ (j) $(\sqrt{5} + \sqrt{2})(\sqrt{5} - \sqrt{2})$
(k) $(\sqrt{7} - \sqrt{13})(\sqrt{7} + \sqrt{13})$ (l) $(2\sqrt{3} + 3\sqrt{2})(2\sqrt{3} - 3\sqrt{2})$
(m) $(1 + \sqrt{3})^2$ (n) $(\sqrt{5} - 2)^2$
(o) $(\sqrt{2} + \sqrt{7})^2$ (p) $(\sqrt{3} - \sqrt{5})^2$

Rationalising the denominator

The denominator is irrational (a surd)

$$\frac{2}{\sqrt{5}} = \frac{2\times\sqrt{5}}{\sqrt{5}\times\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

The denominator is now rational (an integer)

Process

You use $\sqrt{a}\times\sqrt{a} = a$ to get rid of the root sign in the denominator.

Top tip

$$\frac{a}{\sqrt{b}} = \frac{a\times\sqrt{b}}{\sqrt{b}\times\sqrt{b}} = \frac{a\sqrt{b}}{b}$$

root sign no root sign

Example

Express $\frac{2}{\sqrt{18}}$ as a fraction with a rational denominator.

Solution

$$\frac{2}{\sqrt{18}} = \frac{2}{\sqrt{9\times 2}} = \frac{2}{3\sqrt{2}} = \frac{2\times\sqrt{2}}{3\sqrt{2}\times\sqrt{2}} = \frac{2\sqrt{2}}{3\times 2} = \frac{\sqrt{2}}{3}$$

Simplify $\sqrt{18}$

multiply top and bottom by $\sqrt{2}$ to rationalise the denominator

divide top and bottom by 2 (cancel by 2)

Further simplification

If you think of $\sqrt{2}$ as an unknown number like x then you can compare

$$2\sqrt{2} + 3\sqrt{2} = 5\sqrt{2}$$

with $2x + 3x = 5x$

Similarly, by comparing with $x + 3y + 4x - y$ you can simplify

$$\sqrt{3} + 3\sqrt{2} + 4\sqrt{3} - \sqrt{2}$$
$$= \sqrt{3} + 4\sqrt{3} + 3\sqrt{2} - \sqrt{2}$$
$$= 5\sqrt{3} + 2\sqrt{2}$$

Top tip

$\sqrt{2} + \sqrt{3}$ cannot be simplified (compare $x + y$).

Examples

Simplify:

(a) $2\sqrt{5} - 3\sqrt{2} + \sqrt{5} + 4\sqrt{2}$ (b) $\sqrt{2} + \frac{2}{\sqrt{2}}$

Solutions

(a) Rewrite as

$$2\sqrt{5} + \sqrt{5} + 4\sqrt{2} - 3\sqrt{2} = 3\sqrt{5} + \sqrt{2}$$

(compare $2x - 3y + x + 4y$)

(b) $\sqrt{2} + \frac{2}{\sqrt{2}} = \sqrt{2} + \frac{2\times\sqrt{2}}{\sqrt{2}\times\sqrt{2}} = \sqrt{2} + \frac{2\sqrt{2}}{2} = \sqrt{2} + \sqrt{2} = 2\sqrt{2}$

Rationalising Denominators

1. Rationalise the denominators of these fractions:

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{\sqrt{5}}$ (c) $\frac{6}{\sqrt{3}}$ (d) $\frac{8}{\sqrt{2}}$

(e) $\frac{2}{\sqrt{3}}$ (f) $\frac{10}{\sqrt{5}}$ (g) $\frac{7}{\sqrt{3}}$ (h) $\frac{3}{\sqrt{5}}$

(i) $\frac{4}{5\sqrt{2}}$ (j) $\frac{7}{2\sqrt{5}}$

2. Rationalise the denominator of these fractions then simplify:

(a) $\frac{1}{\sqrt{20}}$ (b) $\frac{1}{\sqrt{50}}$ (c) $\frac{10}{\sqrt{12}}$ (d) $\frac{7}{\sqrt{18}}$

3. Write these fractions in their simplest form with a rational denominator:

(a) $\frac{\sqrt{9}}{\sqrt{2}}$ (b) $\frac{\sqrt{5}}{\sqrt{3}}$ (c) $\sqrt{\frac{9}{10}}$ (d) $\sqrt{\frac{3}{5}}$

4. Rationalise the denominators of these fractions and simplify:

(a) $\frac{1}{\sqrt{2}-1}$ (b) $\frac{2}{\sqrt{3}-1}$ (c) $\frac{8}{\sqrt{5}+1}$ (d) $\frac{21}{3-\sqrt{2}}$

(e) $\frac{1}{\sqrt{3}-\sqrt{2}}$ (f) $\frac{2}{\sqrt{7}+\sqrt{2}}$ (g) $\frac{1-\sqrt{3}}{2-\sqrt{5}}$ (h) $\frac{4+\sqrt{5}}{2+\sqrt{3}}$

<u>Indices</u>

The rules of indices

Rule	Comments	Examples
$x^m \times x^n = x^{m+n}$	When multiplying, the indices are added. Note: not in the case $x^m \times y^n$!	$a^2 \times a^3 = a^{2+3} = a^5$
$\dfrac{x^m}{x^n} = x^{m-n}$	When dividing, the indices are subtracted.	$\dfrac{c^7}{c^3} = c^{7-3} = c^4$
$(x^m)^n = x^{mn}$	When raising a power to a power, multiply the indices.	$(y^3)^4 = y^{3\times 4} = y^{12}$
$x^0 = 1$	Any number or expression (other than zero) raised to the power zero gives 1.	$2^0 = 1$ $\quad \left(\dfrac{1}{2}\right)^0 = 1$ $(a+b)^0 = 1$
$x^{-n} = \dfrac{1}{x^n}$	Something to a negative power can be rewritten as 1 divided by the same thing to the positive power.	$a^{-1} = \dfrac{1}{a^1} = \dfrac{1}{a}$ $\quad a^{-3} = \dfrac{1}{a^3}$

Fractional indices

TOP TIP

$a^{\frac{\text{power}}{\text{root}}}$

$9^{\frac{3}{2}}$ — 3: power 3 (cubed); 2: square root

$= \left(\sqrt{9}\right)^3 = 3^3 = 27$

$8^{\frac{2}{3}}$ — 2: power 2 (squared); 3: cube root

$= \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$

Cube roots

$8^{\frac{1}{3}} = \sqrt[3]{8} = 2 \quad (\text{since } 2^3 = 8)$

$27^{\frac{1}{3}} = \sqrt[3]{27} = 3 \quad (\text{since } 3^3 = 27)$

4^{th} Roots

$16^{\frac{1}{4}} = \sqrt[4]{16} = 2 \quad (\text{since } 2^4 = 16)$

$81^{\frac{1}{4}} = \sqrt[4]{81} = 3 \quad (\text{since } 3^4 = 81)$

Note:

$$a^{\frac{m}{n}} \rightarrow \left.\begin{array}{l} \left(a^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{a}\right)^m \\ \left(a^m\right)^{\frac{1}{n}} = \sqrt[n]{a^m} \end{array}\right\} \text{these are the same}$$

Examples

1. Evaluate $16^{\frac{3}{4}} \times 8^{-\frac{4}{3}}$

Solution 1: $16^{\frac{3}{4}} \times \dfrac{1}{8^{\frac{4}{3}}} = \dfrac{\left(\sqrt[4]{16}\right)^3}{\left(\sqrt[3]{8}\right)^4} = \dfrac{2^3}{2^4} = \dfrac{8}{16} = \dfrac{1}{2}$

or solution 2: $(2^4)^{\frac{3}{4}} \times (2^3)^{-\frac{4}{3}} = 2^{\left(4\times\frac{3}{4}\right)} \times 2^{3\times\left(-\frac{4}{3}\right)}$

using $(x^m)^n = x^{mn}$

$= 2^3 \times 2^{-4} = 2^{3+(-4)} = 2^{-1} = \dfrac{1}{2}$

2. Simplify: (a) $x^{\frac{1}{2}} \times 4x^{-\frac{3}{2}}$ $\qquad$ (b) $\left(x^{-\frac{1}{2}}\right)^4$

Solution (a): $4x^{\frac{1}{2}+\left(-\frac{3}{2}\right)} = 4x^{\left(-\frac{2}{2}\right)} = 4x^{-1} = \dfrac{4}{x}$

Solution (b): $x^{-\frac{1}{2}\times 4} = x^{-2} = \dfrac{1}{x^2}$

Preparation for Higher

Indices

1. Simplify each expression.

(a) $a^5 \times a^4$ (b) $n^{-12} \times n^9$ (c) $c^6 \times c$

(d) $d^{\frac{1}{2}} \times d^3$ (e) $3a^4 \times 5a^3$ (f) $4b^9 \times 2b^{-6}$

(g) $8c^8 \times 7c$ (h) $\frac{v^6}{v^2}$ (i) $y^{19} \div y^{-5}$

(j) $\frac{k^8}{k}$ (k) $\frac{12c^5}{6c^3}$ (l) $\frac{48f^{10}}{6f^{-4}}$

(m) $30c^6 \div c^4$ (n) $(c^6)^5$ (o) $(y^7)^{-5}$

(p) $(6h^5)^3$ (q) $(2x^{-2})^5$ (r) $(xy)^5$

(s) $(x^2y^3)^4$ (t) $(h^3k^5)^{-8}$

2. Evaluate:

(a) $25^{\frac{1}{2}}$ (b) $16^{\frac{1}{4}}$ (c) $125^{\frac{1}{3}}$

(d) $128^{\frac{1}{7}}$ (e) $8^{\frac{2}{3}}$ (f) $81^{\frac{3}{4}}$

(g) $1000^{\frac{2}{3}}$ (h) $243^{\frac{3}{5}}$ (i) $625^{-\frac{1}{4}}$

(j) $64^{-\frac{5}{6}}$

3. Simplify:

(a) $k^{\frac{1}{2}} \times k^{\frac{1}{4}}$ (b) $t^{\frac{2}{3}} \times t^{\frac{2}{3}}$ (c) $g^{\frac{3}{4}} \times g^{-\frac{1}{4}}$

(d) $\sqrt[3]{y} \times \sqrt[3]{y}$ (e) $4d^{-\frac{1}{2}} \times 5d^{\frac{3}{2}}$ (f) $2\sqrt[3]{e} \times 4\sqrt[3]{e^2}$

(g) $\frac{d^{\frac{2}{3}}}{d^{\frac{1}{3}}}$ (h) $d^{\frac{3}{4}} \div d^{\frac{1}{4}}$ (i) $\frac{\sqrt[3]{y}}{\sqrt[3]{y}}$

(j) $4d^{\frac{1}{2}} \div 5d^{\frac{3}{2}}$ (k) $\frac{4\sqrt[3]{e}}{2\sqrt[3]{e^2}}$ (l) $\left(4d^{-\frac{1}{2}}\right)^{\frac{3}{2}}$

(m) $(7t)^{-2}$ (n) $\left(c^3d^{\frac{1}{2}}\right)^3$ (o) $(x^4y^2)^{\frac{1}{2}}$

(p) $\left(s^{\frac{1}{2}}t^{\frac{2}{3}}\right)^{\frac{2}{3}}$

Preparation for Higher

Complex Indices

1. Express each fraction as a sum or difference of terms.

(a) $\dfrac{x^6+x^7}{x^3}$ (b) $\dfrac{x^{10}-x^{20}}{x^5}$ (c) $\dfrac{x^2-x^3}{2x^3}$

(d) $\dfrac{2x^4+3x^2}{x^6}$ (e) $\dfrac{x^4-1}{x^2}$ (f) $\dfrac{x^{\frac{2}{3}}+x^{\frac{3}{4}}+2}{x^4}$

(g) $\dfrac{x^4+x^5}{2x^2}$ (h) $\dfrac{x^7+x^2}{3x^4}$ (i) $\dfrac{2\left(x^4-x^6\right)}{4x^2}$

(j) $\dfrac{\sqrt[3]{x}+x^2}{x}$ (k) $\dfrac{2\sqrt{x}+x^2}{\sqrt{x}}$ (l) $\dfrac{6\sqrt[3]{x}+2\sqrt[4]{x^5}}{\sqrt[3]{x^4}}$

2. Express each fraction as a sum or difference of terms.

(a) $\dfrac{(x+2)^2}{x^3}$ (b) $\dfrac{(x+3)(2x-1)}{x}$ (c) $\dfrac{(1-x)^2}{2x}$

(d) $\left(\dfrac{3}{x}-4\right)^2$ (e) $\dfrac{\left(x^2-5\right)\left(x^2+5\right)}{x^2}$

(f) $\left(x+\dfrac{1}{x}\right)\left(x-\dfrac{1}{x}\right)$ (g) $\dfrac{(x-1)^2}{\sqrt{x}}$ (h) $\dfrac{(x+2)^2}{x\sqrt{x}}$

(i) $\left(\dfrac{1}{\sqrt{x}}+\sqrt{x}\right)^2$